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2010 J. Phys. A: Math. Theor. 43 255307

(http://iopscience.iop.org/1751-8121/43/25/255307)

J. Phys. A: Math. Theor. **43** (2010) 255307 (10pp) doi:10.1088/1751-8113/43/25/255307

Spherical designs and anticoherent spin states

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Received 8 March 2010, in final form 28 April 2010
Published 28 May 2010
Online at stacks.iop.org/JPhysA/43/255307

Abstract
Anticoherent spin states are quantum states that exhibit maximally nonclassical behaviour in a certain sense. Any spin state whose Majorana representation is a Platonic solid is called a perfect state. By direct calculation, it has been shown that any perfect state is an anticoherent spin state. We show that any spin state whose Majorana representation is both the orbit of a finite subgroup of $O(3)$ and a spherical t-design must be an anticoherent spin state of order t. Since all Platonic solids are spherical designs, this result gives an explanation of the anticoherence of perfect states and explains their observed order. We also show that any spin state whose Majorana representation lies in any single open hemisphere cannot be anticoherent of any order. This result is then used to give further relations between spherical designs and anticoherent spin states. We also pose some questions relating spherical designs and geometric entanglement.

PACS numbers: 02.10.Ox, 03.65.Fd, 03.67.Mn
Mathematics Subject Classification: 81P99, 81Q99, 52B10, 05B30

1. Introduction

Coherent states have been of interest since their discovery in the 1920s, and provide a crucial connection between classical and quantum behaviour [5]. One can see this connection explicitly by studying the geometry of quantum state space [4]. For any quantum system, the set of coherent states is viewed as an embedding of classical phase space into the space of pure states for that system. The case of spin states is of particular interest and the Majorana representation, introduced as a manifestation of spin alignment in atoms exposed to oscillating magnetic fields, associates spin states with geometric configurations on the sphere [16]. Any spin state whose Majorana representation is a Platonic solid is called a *perfect* state. On the other hand, anticoherent spin states are quantum states first studied in [24] as a class of states that exhibit maximally nonclassical behaviour in a sense made precise here. By direct calculation, it has been shown that any perfect state is an anticoherent spin state, but a number of issues remain unsettled on this class of states.

In this paper, we study anticoherent states from two perspectives: via the Majorana representation and a new set of tools in this setting, that of spherical designs. Specifically, we show that any spin state whose Majorana representation is both the orbit of a finite subgroup of the orthogonal group $O(3)$ and a spherical t-design must be an anticoherent spin state of order t. Since all Platonic solids are spherical designs, this result gives an explanation of the anticoherence of perfect states and explains their observed order. We also show that any spin state whose Majorana representation lies in any single open hemisphere cannot be anticoherent of any order. This result is then used to give further relations between spherical designs and anticoherent spin states. We also pose some questions relating spherical designs and geometric entanglement.

The paper is arranged as follows. The next three sections include the basics on anticoherent spin states, the Majorana representation and spherical designs. The subsequent two sections present the derivation of our main results, and the final section includes an application to entanglement theory.

2. Anticoherent spin states

An important example of coherent states are the coherent *spin* states. For any s, they form a sphere of radius $\sqrt{s/2}$ inside $\mathbb{C}\mathbf{P}^{2s}$ (spin space). This 2-sphere serves as the classical phase space for spin angular momentum [4].

As useful as this makes the coherent states, one could argue that today we are more interested in states that *do not* mimic classical behaviour. With this in mind, *anticoherent* spin states were introduced to serve as the 'opposite' of coherent spin states [24]. As such they should be as 'far away' as possible from the classical phase space embedding, hopefully making them useful for applications involving non-classical phenomena. They are defined as the states $|\psi\rangle$ for which $\langle \mathbf{n}\cdot\mathbf{S}\rangle = \langle\psi|\mathbf{n}\cdot\mathbf{S}|\psi\rangle$ is independent of $\mathbf{n}$, where $\mathbf{S} = (S_x, S_y, S_z)$ is the usual spin operator and $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$. In other words, the expected value of a measurement of spin in an anticoherent state is the same for any direction we choose to measure. The canonical example is $|\psi\rangle = |s = 1, m_z = 0\rangle$. This state satisfies $\langle\mathbf{S}\rangle = \mathbf{0}$, which is equivalent to the definition. Indeed, matrix representations for the spin operators [11, p 195] in the ordered basis $\{|1,-1\rangle, |1,0\rangle, |1,1\rangle\}$ are given (up to normalization) by

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad S_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix},$$

and it is easy to see that $\langle\psi|S_j|\psi\rangle = 0$ for $j = x, y, z$.

Directional invariance of the measurement is useful; however, suppose one computed other statistics of $\mathbf{n}\cdot\mathbf{S}$ in the state $|1, 0\rangle$, such as the variance. One may see a directional dependence even though $\langle 1, 0|(\mathbf{n}\cdot\mathbf{S})|1, 0\rangle$ is independent of $\mathbf{n}$. This is due to the directional dependance of $\langle 1, 0|(\mathbf{n}\cdot\mathbf{S})^2|1, 0\rangle$, the second of moment spin along $\mathbf{n}$. We can remove this directional signature by requiring that higher moments of $\mathbf{n}\cdot\mathbf{S}$ be direction independent.

Definition 1. *We say that a state $|\psi\rangle$ is anticoherent to order t if $\langle(\mathbf{n}\cdot\mathbf{S})^k\rangle$ is independent of $\mathbf{n}$ for $k = 1, \ldots, t$.*

One can readily verify by direct calculation that the state $|\psi\rangle = |s = 1, m_z = 0\rangle$ is anticoherent to order 1.

3. The Majorana representation

The Majorana representation [16] works as follows. Every spin-s state

$$|\psi\rangle = \sum_{m=-s}^{s} a_m |m\rangle$$

is mapped (bijectively) to a polynomial of degree $2s$:

$$M_{|\psi\rangle}(z) = \sum_{m=-s}^{s} (-1)^{m-s} a_m \sqrt{\binom{2s}{s+m}} z^{s+m}.$$

$M_{|\psi\rangle}(z)$ has $2s$ (not necessarily distinct) roots in the complex plane, generating an unordered set of $2s$ points on the Riemann sphere via stereographic projection from the south pole: $v : \mathbb{C} \cup \{\infty\} \to \mathbb{R}^3$,

$$v(z) = \frac{1}{|z|^2+1}(2\Re(z), 2\Im(z), |z|^2 - 1), \qquad v(\infty) = (0, 0, -1).$$

From this seemingly simple correspondence we obtain many useful geometric characterizations. For instance, coherent states are represented as a single point on the sphere with the multiplicity $2s$. Eigenstates of $\mathbf{n} \cdot \mathbf{S}$ with the eigenvalue m are represented by $s+m$ points at $\mathbf{n}$ and $s-m$ points at the antipode $-\mathbf{n}$. Rotation operators are given by rotations of the sphere and the operation of time reversal is given by an inversion of the sphere through the origin. Remarkably, Zimba and Penrose used this representation to prove Bell's non-locality theorem *without* probabilities [25].

Among the numerous applications, this representation actually motivated the definition of anticoherent spin states. Since coherent states are given by a single point on the sphere, it was thought that anticoherent states would be uniform distributions—the Platonic solids being of primary interest. It was shown that 'Platonic' spin states are indeed anticoherent [24], but the question of why still remained open. Here we answer that question and propose many others by relating anticoherent spin states to spherical designs.

4. Spherical designs

We begin this section by introducing some terminology. In what follows $\Omega_n = \{x \in \mathbb{R}^n : \|x\| = 1\}$ will denote the unit sphere in $\mathbb{R}^n$ endowed with the standard measure. A polynomial p in n variables is said to be homogenous of degree d if $p(kx) = k^d p(x)$ for all $k \in \mathbb{R}$ and $x \in \mathbb{R}^n$. A polynomial p in n variables is said to be harmonic if it satisfies Laplace's equation on $\mathbb{R}^n$ (i.e. $\Delta p(x) = 0$). If G is a finite subgroup of $O(n)$, then the G-orbit of a point $x \in \mathbb{R}^n$ is the set $\{gx : g \in G\}$ and a G-invariant polynomial is a polynomial p in n variables for which $p(gx) = p(x)$ for all $g \in G$ and all $x \in \mathbb{R}^n$. We note that a simple chain rule argument (see for example page three of [2] for details) shows that if $T \in O(n)$ and $f : \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable then $\Delta(f \circ T) = (\Delta f) \circ T$. It follows that if G is a finite subgroup of $O(n)$ and p is a G-invariant polynomial, then Δp is also a G-invariant polynomial. Finally we recall that the trace of a square matrix A denoted $\mathrm{tr}(A)$ is the sum of the entries on the main diagonal of A.

Definition 2. *A spherical t-design on Ω_n is a finite set of points $x_1, x_2, \ldots, x_m \in \Omega_n$ such that $\frac{1}{\mathrm{vol}(\Omega_n)} \int_{\Omega_n} p(x)\,\mathrm{d}x = \frac{1}{m}\sum_{i=1}^{m} p(x_i)$ for all n-variable homogeneous polynomials $p(x)$ of degree less than or equal to t.*

Spherical designs were first introduced in [8].

Theorem 1 (Goethals–Seidel) [10]. *Let G be a finite subgroup of $O(n)$. Then every G-orbit is a spherical t-design if and only if there are no G-invariant nonconstant n-variable homogeneous harmonic polynomials with degree less than or equal to t.*

This result motivates the following definition.

Definition 3. *Let G be a finite subgroup of $O(n)$. We say that G is t-homogeneous if every G-orbit is a spherical t-design.*

We will be particularly interested in spherical t-designs in $\mathbb{R}^3$. The vertices of a regular polytope form an important example of spherical t-designs. We note that the finite subgroups of $O(3)$ have been characterized (see [7] or [22]). It is known that the vertex sets of all of the regular polyhedra are orbits of the finite subgroups of $SO(3)$. The tetrahedron is the orbit of a two-homogeneous group, the cube and the octahedron each are the orbit of a three-homogeneous group and the dodecahedron and the icosahedron each are the orbit of a five-homogeneous group [3]. The dihedral group of order $2n$ is another finite subgroup of $SO(3)$ which is one-homogeneous.

The following modified version of the Goethals–Seidel theorem will be useful to us.

Theorem 2. *Let G be a finite subgroup of $O(n)$. Then G is t-homogeneous if and only if the only G-invariant n-variable homogeneous polynomials with degree less than or equal to t are all of the form $c\left(x_1^2+x_2^2+\cdots+x_n^2\right)^q$, where $c \in \mathbb{R}$ and q is an integer with $0 \leqslant 2q \leqslant t$.*

Proof. Since $c\left(x_1^2+x_2^2+\cdots+x_n^2\right)^q$ is never a harmonic polynomial if c is nonzero, the if direction of this result is an immediate consequence of the Goethals–Seidel theorem. Now suppose G is t-homogeneous. Let p be a homogeneous G-invariant polynomial of degree $2j-1 \leqslant t$. We will show that p is zero by induction on j. If $j=1$, p is linear and hence harmonic. Therefore, p must be zero by the Goethals–Seidel theorem. Now suppose p has degree $2j-1 \leqslant t$; then Δp is a homogeneous G-invariant polynomial of degree $2j-3$ and hence must be zero by the induction hypothesis. Then p is harmonic and therefore must be zero by the Goethals–Seidel theorem. Now let p be a homogeneous G-invariant polynomial of degree $2j \leqslant t$, we will show that p must be of the form $c\left(x_1^2+x_2^2+\cdots+x_n^2\right)^j$, also using induction on j. Any homogeneous n-variable polynomial p of degree 2 can be written as $p(x)=x^TAx$, where A is an $n\times n$ real symmetric matrix, p will be harmonic if and only if the trace of A is zero. Since $p(x)=r(x)+\frac{\operatorname{tr}(A)}{n}\left(x_1^2+x_2^2+\cdots+x_n^2\right)$ where $r(x)=x^T\left(A-\frac{\operatorname{tr}(A)}{n}I\right)x$ is a harmonic homogeneous polynomial of degree 2. If p is G-invariant, then so is r which means that r is zero which proves the base case of our induction. Now suppose p has degree $2j\leqslant t$, then Δp is a homogeneous G-invariant polynomial of degree $2j-2$ and hence must be of the form $c\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{j-1}$. Then p is the sum of a polynomial of the form $c\left(x_1^2+x_2^2+\cdots+x_n^2\right)^j$ and a G-invariant degree $2j$ harmonic polynomial; since the latter is zero our result follows. □

5. Main theorem

As has been noted by many authors, applying a rotation operator to a spin state causes the points in its Majorana representation to rotate rigidly. (See for instance [12, 24] or chapter 7 of [4].) We phrase this observation slightly differently as follows.

Lemma 1. *Let $R \in SO(3)$. Then there is a unitary linear operator L on the spin-s state space with the property that if $|\phi\rangle$ has a Majorana representation corresponding to the points*

$\{z_m\}_{m=1}^{2s}$ on the Riemann sphere, then $|L\phi\rangle$ is a state whose Majorana representation consists of the points $\{Rz_m\}_{m=1}^{2s}$.

Lemma 2. *Let $A \in O(3)$. Let $|\phi\rangle$ be the state whose Majorana representation consists of the points $\{z_m\}_{m=1}^{2s}$ on the Riemann sphere and let $|\psi\rangle$ be the state whose Majorana representation consists of the points $\{Az_m\}_{m=1}^{2s}$. Then $\langle\phi|(\mathbf{n}\cdot\mathbf{S})^k|\phi\rangle = \langle\psi|(A\mathbf{n}\cdot\mathbf{S})^k|\psi\rangle$ for all k.*

Proof. First suppose $A \in SO(3)$. We note that this result holds if $|\phi\rangle$ is an eigenstate of $\mathbf{n}\cdot\mathbf{S}$. We can then extend to the general $|\phi\rangle$ using linearity and the previous lemma. We can extend this result to $O(3)$ by proving that it holds when A is a single non-trivial reflection such as the reflection through the x–z plane. It can be seen that this reflection is induced by the conjugate linear operator which maps $\sum_{m=-s}^{s} a_m|m\rangle \mapsto \sum_{m=-s}^{s} \overline{a_m}|m\rangle$. This operator commutes with S_x and S_z but anticommutes with S_y and our result follows. □

We can now state our main theorem.

Theorem 3. *Let G be a t-homogeneous finite subgroup of $O(3)$. Then any state whose Majorana representation is a G-orbit is an anticoherent spin state of order at least t.*

Proof. We note that for any fixed state, $f(\mathbf{n}) = \langle(\mathbf{n}\cdot\mathbf{S})^k\rangle$ is a degree k homogeneous polynomial in n_x, n_y and n_z. Furthermore it follows from the previous lemma that if the state has a Majorana representation which is a G-orbit, then f is G-invariant. By theorem 2, whenever $k \leqslant t$, $\langle(\mathbf{n}\cdot\mathbf{S})^k\rangle$ must be zero if k is odd or is of the form $c\left(n_x^2+n_y^2+n_z^2\right)^{\frac{k}{2}}$ for $c \in \mathbb{R}$ if k is even. In either case $\langle(\mathbf{n}\cdot\mathbf{S})^k\rangle$ is independent of the unit vector $\mathbf{n}$. □

Remark 1. This result explains why any state whose Majorana representation is a Platonic solid is an anticoherent spin state. As was noted in [24], a tetrahedral state is anticoherent of order 2, and an octahedral state is anticoherent of order 3. The states corresponding to the cube, dodecahedron and icosahedron can now be seen to be anticoherent of orders 3, 5 and 5, respectively.

6. Equivalence in lower dimensions

There are other connections between spherical designs and anticoherent spin states. Zimba [24] has shown that there are no anticoherent spin states of order 2 for spins $\frac{1}{2}$, 1 and $\frac{3}{2}$. This was improved in [15] where it was shown that the only spins for which there are no anticoherent spin states of order 2 are $\frac{1}{2}$, 1, $\frac{3}{2}$ as well as $\frac{5}{2}$. It was proved by Mimura [20] that in three dimensions there are n-point spherical 2-designs if and only if n is not equal to one of 1, 2, 3, 5.

We note that the points of a spherical 1-design cannot lie in a common open hemisphere. In this section, we will prove the same for the Majorana representation of an anticoherent state of order 1. This result will allow us to prove that a state with spin 1 or $\frac{3}{2}$ is anticoherent of order 1 if and only if its Majorana representation is a spherical 1-design.

We first need the following result which is known as the Cohn–Egerváry–Szegő theorem (named after the three mathematicians who independently discovered this result in 1922).

Lemma 3. *Let $f = \sum_{j=0}^{n} a_j z^j$ be an nth degree polynomial all of whose roots lie in the open unit disc on the complex plane and $g = \sum_{j=0}^{n} b_j z^j$ be an nth degree polynomial all of whose roots lie in the closed unit disc. Then the Schur–Szegő composition of f and g defined as $f \star g = \sum_{j=0}^{n} \frac{a_j b_j}{\binom{n}{j}} z^j$ also has all of its roots in the open unit disc.*

The original references to this result are [6, 9, 23]. This result can also be found in the standard references for the theory of polynomials. (It is listed as corollary 16.1a of [17] and as theorem 3.4.1e of [21].)

Proposition 1. *Any spin-s state whose Majorana representation consists of points lying entirely in an open hemisphere of the Riemann sphere cannot be anticoherent of order 1.*

Proof. Because of lemma 2, we may assume that the Majorana representation of $|\psi\rangle = \sum_{m=-s}^{s} a_m|m\rangle$ has all of its points strictly below the x–y plane. This means that the polynomial $M_{|\psi\rangle}(z) = \sum_{m=-s}^{s}(-1)^{m-s}a_m\sqrt{\binom{2s}{s+m}}z^{s+m}$ has all of its roots in the open unit disc. Now the polynomial $q(z) = (z-1)(z+1)^{2s-1} = \sum_{m=-s}^{s}\binom{2s}{m}\frac{m}{s}z^{s+m}$ has all of its roots in the closed unit disc. By lemma 3, $p(z) = \overline{M}_{|\psi\rangle}(z) \star q(z) \star M_{|\psi\rangle}(z) = \sum_{m=-s}^{s}\frac{m}{s}|a_m|^2z^{s+m}$ has all of its roots in the open unit disc. Therefore, $\frac{1}{s}\langle S_z\rangle = p(1)$ must be nonzero and hence $\langle(\mathbf{n}\cdot\mathbf{S})\rangle$ is not the zero polynomial. Hence $|\psi\rangle$ cannot be anticoherent of order 1. □

We note that if two points on a sphere are not antipodal then they must lie in a common open hemisphere; hence, the 'only if' direction of the following result is a direct corollary.

Proposition 2. *A spin-1 state is anticoherent to order 1 if and only if its Majorana representation is a spherical 1-design; i.e. the Majorana roots are antipodal on the Riemann sphere.*

We note that the if direction follows from direct calculation. We may prove a similar result for spin-$\frac{3}{2}$ systems. In what follows we take $\hbar = 1$. Recall that $v(z)$ is the stereographic projection of z onto the Riemann sphere described in section 3.

Lemma 4. *Let* $s = \frac{3}{2}$ *and* $|\psi\rangle = \sum_{m=-\frac{3}{2}}^{\frac{3}{2}} a_m|m\rangle$ *be a state with* $a_{\frac{3}{2}} \neq 0$. *Then*

$$\langle\mathbf{n}\cdot\mathbf{S}\rangle = \frac{|a_{\frac{3}{2}}|^2}{2}\prod_{i=1}^{3}(1+|r_i|^2)\sum_{i=1}^{3}c_i(\mathbf{n}\cdot v(r_i)),$$

where

$$c_i = 1 - \frac{1}{3}\frac{|r_{i\oplus1} - r_{i\oplus2}|^2}{(1+|r_{i\oplus1}|^2)(1+|r_{i\oplus2}|^2)},$$

r_i *are the roots of* $M_{|\psi\rangle}(z)$ *and* $\oplus$ *is addition mod 3.*

Proof. The spin operator in question is up to normalization [11, p 195]

$$\mathbf{n}\cdot\mathbf{S} = \frac{1}{2}\begin{bmatrix} 3n_z & \sqrt{3}(n_x - \mathrm{i}n_y) & 0 & 0 \\ \sqrt{3}(n_x + \mathrm{i}n_y) & n_z & 2(n_x - \mathrm{i}n_y) & 0 \\ 0 & 2(n_x + \mathrm{i}n_y) & -n_z & \sqrt{3}(n_x - \mathrm{i}n_y) \\ 0 & 0 & \sqrt{3}(n_x + \mathrm{i}n_y) & -3n_z \end{bmatrix}.$$

Representing our state with Viète's formulae for $M_{|\psi\rangle}(z)$ we can write $\langle\mathbf{n}\cdot\mathbf{S}\rangle$ compactly as

$$\frac{\langle\mathbf{n}\cdot\mathbf{S}\rangle}{|a_{\frac{3}{2}}|^2} = f_x(r_1, r_2, r_3)n_x + f_y(r_1, r_2, r_3)n_y + f_z(r_1, r_2, r_3)n_z,$$

where $f_i = f_i(r_1, r_2, r_3)$ are the functions of the roots of $M_{|\psi\rangle}(z)$ for $i = x, y, z$. It follows that

$$\begin{aligned}
f_x(r_1, r_2, r_3) &= \sum_{i=1}^{3} \Re(r_i)\left(1 + \frac{2}{3}(|r_{i\oplus 1}|^2 + |r_{i\oplus 2}|^2 + \Re(r_{i\oplus 1}\overline{r_{i\oplus 2}})) + |r_{i\oplus 1}|^2|r_{i\oplus 2}|^2\right) \\
&= \sum_{i=1}^{3} \Re(r_i)\left((1 + |r_{i\oplus 1}|^2)(1 + |r_{i\oplus 2}|^2) - \frac{1}{3}|r_{i\oplus 1} - r_{i\oplus 2}|^2\right) \\
&= \frac{1}{2}\prod_{i=1}^{3}(1 + |r_i|^2)\sum_{i=1}^{3}\frac{2\Re(r_i)}{1 + |r_i|^2}\left(1 - \frac{1}{3}\frac{|r_{i\oplus 1} - r_{i\oplus 2}|^2}{(1 + |r_{i\oplus 1}|^2)(1 + |r_{i\oplus 2}|^2)}\right) \\
&= \frac{1}{2}\prod_{i=1}^{3}(1 + |r_i|^2)\sum_{i=1}^{3} c_i v_x(r_i).
\end{aligned}$$

Similar results hold for f_y and f_z, hence

$$\langle \mathbf{n}\cdot\mathbf{S}\rangle = \frac{\left|a_{\frac{3}{2}}\right|^2}{2}\prod_{i=1}^{3}(1 + |r_i|^2)\sum_{i=1}^{3} c_i(\mathbf{n}\cdot v(r_i)).$$
□

We note that $c_i = 1 - \frac{1}{12}[d(r_{i\oplus 1}, r_{i\oplus 2})]^2$ where d is the chordal metric ($d(r_{i\oplus 1}, r_{i\oplus 2})$ is the distance between $v(r_{i\oplus 1})$ and $v(r_{i\oplus 2})$ in $\mathbb{R}^3$). Since the maximum value of the chordal metric is 2, the c's are always positive.

Proposition 3. *A spin-$\frac{3}{2}$ state is anticoherent to order 1 if and only if its Majorana representation is a spherical 1-design; i.e. the Majorana roots are equally spaced on a great circle of* Ω_3.

Proof. The if direction may be verified by direct calculation so we prove the only if direction. $\langle \mathbf{n}\cdot\mathbf{S}\rangle$ is independent of $\mathbf{n}$ if and only if $c_1 v(r_1) + c_2 v(r_2) + c_3 v(r_3) = 0$. For this to occur, $v(r_1)$, $v(r_2)$ and $v(r_3)$ must be linearly dependent which means that they lie on the same plane through the origin and hence on the same great circle. (We note that this statement also follows from the fact that any three points on a sphere which do not lie on a single great circle must lie in a common open hemisphere.) Since the c_i's depend only on the chordal metric and are rotationally invariant, we may assume that $v(r_1)$, $v(r_2)$ and $v(r_3)$ lie on the equator. So let $v(r_k) = (\cos(\theta_k), \sin(\theta_k), 0)$ for $k = 1, 2, 3$. For ease of notation, we will let $\alpha_k = |\theta_{k\oplus 1} - \theta_{k\oplus 2}|$. Then using the law of cosines, $c_k = \frac{4}{6} + \frac{1}{6}\cos^2(\alpha_k) = \frac{5}{6} - \frac{1}{6}\sin^2(\alpha_k)$. Because $c_1 v(r_1) + c_2 v(r_2) + c_3 v(r_3) = 0$, there is a triangle with side lengths $\{c_k\}_{k=1}^3$; the sine of the angle opposite the side of length c_k is $\sin(\alpha_k)$. The law of sines states that $\frac{\sin(\alpha_1)}{c_1} = \frac{\sin(\alpha_2)}{c_2} = \frac{\sin(\alpha_3)}{c_3}$. Now using our expression for c_k, we get $\frac{\sin(\alpha_k)}{c_k} = \frac{6\sin(\alpha_k)}{5-\sin^2(\alpha_k)} = f(\sin(\alpha_k))$, where $f(x) = \frac{6x}{5-x^2}$. Since $f(x)$ is strictly increasing on [0, 1], $f(\sin(\alpha_1)) = f(\sin(\alpha_2)) = f(\sin(\alpha_3))$ implies that $\sin(\alpha_1) = \sin(\alpha_2) = \sin(\alpha_3)$. Hence either the sines are all zero or all alphas are $\frac{2\pi}{3}$; the former would violate the equation $c_1 v(r_1) + c_2 v(r_2) + c_3 v(r_3) = 0$ and the latter corresponds to equally spaced points around the great circle. □

In light of the results in both this section and the previous one, several questions arise about the relation between spherical designs and anticoherent spin states. The most general conjecture is the following.

Conjecture 1. *Any spin state is anticoherent of order t if and only if its Majorana representation is a spherical t-design.*

7. Application: symmetric state entanglement

It is well known that a symmetric pure state of n-qubits can be written as

$$|\psi_{\mathrm{sym}}\rangle = C \sum_{\sigma\in S_n} |r_{\sigma(1)}\rangle|r_{\sigma(2)}\rangle\cdots|r_{\sigma(n)}\rangle,$$

where $|r_i\rangle$ are the qubits, C is a normalization constant and S_n is the symmetric group of n elements [16, 18]. Amalgamating the representations of each qubit onto the same sphere, we obtain a Majorana representation $\{v(r_i)\}_{i=1}^n$ of $|\psi_{\mathrm{sym}}\rangle$. For example, the state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ has the decomposition

$$|r_1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |r_2\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + \mathrm{e}^{\mathrm{i}2\pi/3}|1\rangle), \qquad |r_3\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + \mathrm{e}^{\mathrm{i}4\pi/3}|1\rangle)$$

giving an equilateral triangle on the equator of the Riemann sphere.

Recently, the preferred measure of entanglement for symmetric pure states has been the geometric measure [13, 14], defined as

$$E_G(|\psi\rangle) = 1 - \max_{|\phi\rangle\in\mathrm{Prod}} |\langle\psi|\phi\rangle|^2.$$

The advantage of this measure is that the product state maximizing the overlap with $|\psi\rangle$ can be found in the symmetric subspace when $|\psi\rangle$ is itself symmetric, drastically simplifying the computation of E_G [14]. This allows us to phrase the geometric measure of entanglement in terms of the Majorana representation.

Proposition 4. *Let* $|\psi\rangle \in \mathrm{Sym}((\mathbb{C}^2)^{\otimes n})$ *be a pure state with Majorana representation* $\{v(r_i)\}_{i=1}^n$. *Then*

$$E_G(|\psi\rangle) = 1 - \frac{C^2(n!)^2}{4^n} \max_{\mathbf{n}\in\Omega_3}\left(\prod_{i=1}^n \|v(r_i) - \mathbf{n}\|^2\right).$$

Proof. Let $|\phi\rangle = \frac{1}{\sqrt{1+|r_\phi|^2}}(|0\rangle + r_\phi|1\rangle) \in \mathbb{C}^2$ with $r_\phi \neq 0$. Note that r_ϕ is the Majorana root of $|\phi\rangle$ when viewed as a spin-1/2 state. Writing each $|r_i\rangle$ in the expansion of $|\psi\rangle$ in a similar fashion as $\frac{1}{\sqrt{1+|r_i|^2}}(|0\rangle + r_i|1\rangle)$, one can easily verify that

$$|\langle r_i|\phi\rangle|^2 = \frac{|r_i - r_\phi^*|^2}{(1+|r_i|^2)(1+|r_\phi^*|^2)} = \frac{1}{4}d(r_i, r_\phi^*)^2,$$

where d is the chordal metric and $r_\phi^* = -1/\bar{r_\phi}$ is the antipode of r_ϕ, i.e. the root corresponding to the opposite point on the sphere. Thus, for any symmetric product state $|\Phi\rangle = |\phi\rangle\otimes\cdots\otimes|\phi\rangle$, we have $|\langle\psi|\Phi\rangle|^2 = \frac{C^2(n!)^2}{4^n}\prod_{i=1}^n \|v(r_i) - v(r_\phi^*)\|^2$ from which it follows that

$$E_G(|\psi\rangle) = 1 - \frac{C^2(n!)^2}{4^n} \max_{\mathbf{n}\in\Omega_3}\left(\prod_{i=1}^n \|v(r_i) - \mathbf{n}\|^2\right).$$

If $r_\phi = 0$ or ∞, the result follows by rotational invariance. □

The Majorana representation of symmetric product states consists of a single point with the multiplicity n (equivalent to that of coherent spin states) and so $C = \frac{1}{n!}$ and $v(r_i) = v$ for all i, hence E_G achieves its minimum value of zero. If anticoherent states are the 'opposite' of coherent states, it would be interesting to investigate the geometric entanglement of their symmetric counterparts. In particular, what can be said about the entanglement of a state represented by a spherical t-design?

Recently, it has been shown that symmetric states with the highest geometric entanglement for four and six qubits are represented respectively by the tetrahedron, a spherical 2-design, and the octahedron, a spherical 3-design [1, 19]. Throughout the analysis many other states represented by spherical designs such as the equilateral triangle and the triangular bipyramid were also shown to be considerably entangled. This evidence suggests that symmetric states whose Majorana representations are spherical t-designs would be strong candidates for applications involving large degrees of geometric entanglement.

Acknowledgments

We would like to thank the referee for many helpful suggestions and for bringing the authors' attention to [1, 19]. Part of this work was completed during the Fields Institute 2009 Thematic Program on Mathematics in Quantum Information. JC was supported by an NSERC Undergraduate Summer Research Award. RP was supported by NSERC Discovery grant 400096 and a University of Guelph start-up grant. DWK was supported by NSERC Discovery grant 400160, NSERC Discovery Accelerator Supplement 400233 and Ontario Early Researcher Award 048142.

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